

# NON-LINEAR DYNAMICS AND CHAOS CONTROL OF A PHYSICAL PENDULUM WITH A ROTATING MASS

S. A. Zareiyan Jahromi<sup>1</sup>, A. H. Haji<sup>2</sup>, M. Mahzoon<sup>3</sup>

Shiraz University, Department of Mechanical Engineering, Mola Sadra St., Shiraz, Iran  
E-mail: alizareiyan@yahoo.com

## Abstract

The dynamic behavior of a physical pendulum system with a rotating mass at the end is studied in this paper. By applying various procedures such as phase portrait, Poincaré map, and bifurcation diagram, a variety of periodic solutions are studied and the phenomenon of the chaotic motion is presented. The effect of changing parameters in the system could be found in the bifurcation diagram. Further, chaotic motion can be verified by using Lyapunov exponent. Besides, non-feedback control, delayed feedback control, and adaptive control are used to control chaos effectively.

**Keywords:** Chaos – Chaos Control – Lyapunov exponent – Bifurcation

## Introduction

The dynamics of two rigid bodies pinned together, in the surface and with a fixed point, have two degrees of freedom. If one rotation is prescribed, the degree of freedom reduces to one, yet the set of kinematically possible motions is still quite rich. The motion of such a physical pendulum with a rotating mass at the end will be considered in this paper, when there are motor and damping torques other than the gravity force.

Most of the physical systems are non-linear in nature, and are described by non-linear equations of motion. A common example is a physical pendulum. Through extensive analytical investigations, analogue and numerical simulations, as well as experimental observations, it has been shown that a pendulum exhibits a rich variety of non-linear bifurcational phenomena [1,2]. In practice, many works have been done on how point equilibria, periodic and chaotic attractors are created, changed or destroyed as system parameters are varied.

A number of numerical results such as phase portraits, Poincaré maps, bifurcation diagrams, and Lyapunov exponents [3] are used to study the dynamical behavior of physical pendulum system. The phase portrait is a collection of trajectories that

represent the solution of differential equations of motion in the phase space. The more informative representation of the periodic solutions is the Poincaré map method. The bifurcation is the special phenomena in a non-linear system. As the parameters are changed in the non-linear dynamical system, the characteristics of the equilibrium points change suddenly. Chaotic motion is the motion that has a sensitive dependence on initial conditions in the deterministic non-linear physical systems. The Lyapunov exponent test is a powerful method to measure the sensitivity of the dynamical system to changes in initial conditions.

Various methods for control of chaos are presented. In practice, it is often desired that chaos can be avoided and that the system performance will be improved or changed in some way. Clearly, the ability to control chaos, that is to convert chaotic oscillations into desired regular ones with a periodic time dependence, would be beneficial in working with a particular system. For this purpose, non-feedback control, delayed feedback control, and adaptive control are used to control chaos. As a result, the chaotic system can be controlled effectively.

---

<sup>1</sup> - M.Sc. student, Department of Mechanical Engineering, Shiraz University

<sup>2</sup> - PhD student, Department of Mechanical Engineering, Shiraz University

<sup>3</sup> - PhD, Department of Mechanical Engineering, Shiraz University

## Equation of motion

The pendulum device shown in Fig. 1 consists of a physical pendulum and a rotating mass at the end. The motor torque produces an angular acceleration of the end mass, which generates a coupling torque at the pendulum axes [4]. Also, damping effects is considered by a multiple of third power of angular velocities. The angles  $q_1$  and  $q_2$  will be taken as generalized co-ordinates (Fig. 1).

The dynamical equations of the model can be written in standard form as

$$\begin{bmatrix} m_1 + m_2 & -m_1 \\ m_1 & m_1 \end{bmatrix} \ddot{q} + D = \begin{bmatrix} mgl \sin(q_1) \\ u \end{bmatrix}. \quad (1)$$

The change of coordinates  $x_q = [x_{q1}, x_{q2}]^T = [q_1, q_1 + q_2]^T$  leads to a simplified description.

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix} \ddot{x}_q + \begin{bmatrix} -m_3 \sin(x_{q1}) \\ 0 \end{bmatrix} + \bar{D} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} u, \quad (2)$$

where  $m_3 = mgl$ ; and  $D_{2*1}$  and  $\bar{D}_{2*1}$  are damping terms.

By considering  $u$  as  $u = e\dot{x}_{q1}x_{q1}^2 + a \sin(\omega t)$ , where  $a$  and  $e$  are parameters and  $\omega$  is the frequency of motor torque, two rows of equation (2) are decoupled, and the first one that will be analyzed in this paper is

$$\begin{aligned} m_2 \ddot{x}_{q1} - m_3 \sin(x_{q1}) + c\dot{x}_{q1}^3 + \\ e\dot{x}_{q1}x_{q1}^2 + a \sin(\omega t) = 0, \end{aligned} \quad (3)$$

where  $c$  is the damping coefficient.

If  $x = x_{q1}$ ,  $y = x_{q2}$ , equation (3) produces a non-autonomous form in the state space.

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m_2} (m_3 \sin(x) - cy^3 - eyx^2 - a \sin(\omega t)) \end{cases} \quad (4)$$

By describing  $x = x_{q1}$ ,  $y = x_{q2}$ ,  $z = \omega t$ , the autonomous state space form of equation (3) will be

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m_2} (m_3 \sin(x) - cy^3 - eyx^2 - a \sin(z)) \\ \dot{z} = \omega \end{cases} \quad (5)$$

In spite of dimensional difference of equations (4) and (5), they show similar response, and in this paper, both of them are used in the numerical analysis of the nonlinear system.

## Phase Portraits and Poincaré Map

The evolution of a set of trajectories starting from various initial conditions is presented in the phase space. When the solution becomes stable, the asymptotic behavior of the phase space trajectories are particularly interesting and transient behaviors in

the system are neglected. The phase portraits of the physical pendulum system, equation (4), are plotted in Fig.2.

The technique introduced by Poincaré deals with the question of the three-dimensional phase space  $(\beta, \dot{\beta}, t)$  whenever  $t$  is a multiple of  $T = 2\pi/\omega$ . Here  $T$  is the period of the external torque. The Poincaré section is constructed by viewing the phase space diagram stroboscopically in such way that the motion is observed periodically. By using MATLAB numerical integration tools, the solutions of the physical pendulum system obtained by Poincaré maps are shown in Fig.2, which can be compared with the phase portraits.

Fig.2 (a), (b), and (c) show phase portraits and Poincaré maps of the non-linear system of equation (4), if constants are chosen as

$$\begin{cases} m_1 = 2, m_2 = 1, m_3 = 10, \\ c = 0.01, \\ e = 0.2, \\ a = 17, \end{cases} \quad (6)$$

Period 1-T, chaotic attractor, and period 3-T are shown in different parts of Fig.2 for variant motor torque frequency.

## Bifurcation Diagram and Lyapunov Exponent

The information about the dynamics of a non-linear system for specific values of parameters is provided. The dynamics may be viewed more completely over a range of parameter values. As the parameter is changed, the periodic solutions are created, destroyed, or their stability may be changed.

The bifurcation diagrams of the non-linear system of equation (4) and conditions (6) are plotted in Fig.3 using MATLAB numerical integration methods. At each  $u$  the points of Poincaré map in the transient state of motion are neglected.

The Lyapunov exponent may be used to show the sensitive dependence on initial conditions. Thus, if  $d_0$  is a measure of initial distance between the two starting points, their subsequent distance grows as  $d(\tau) = d_0 2^{\lambda \tau}$ , where  $\lambda$  is the Lyapunov exponent.

The divergence of chaotic orbits can only be locally exponential, because if the system is bounded,  $d(\tau)$  cannot grow to infinity. A measure of this divergence is that the exponential growth at many points along a trajectory has to be averaged. Thus Lyapunov exponent can be expressed as

$$\lambda = \frac{1}{\tau_N - \tau_0} \sum_{k=1}^N \log_2 \frac{d(\tau_k)}{d_0(\tau_k - 1)}. \quad (7)$$

The signs of Lyapunov exponents provide a quantitative picture of a dynamical system [3]. The criterion is

$$\begin{aligned} \lambda > 0 & \quad (\text{Chaotic}), \\ \lambda \leq 0 & \quad (\text{Regular motion}) \end{aligned}$$

The Lyapunov exponents of the non-linear dynamical system, equation (5) are plotted in Fig.4

The bifurcation diagram provides a summery of the essential dynamics and is therefore a useful way to observe non-linear dynamical behavior. To investigate bifurcation further, the phase portraits, Poincaré maps, and power spectra are used. The periodic and chaotic motions could be distinguished by the bifurcation diagram, but the quasiperiodic motion and chaotic motion may be confused. However, they can be distinguished by the Lyapunov exponent method.

Fig. 2, 3, and 4 provide similar results, for example about existence of chaotic attractor for the non-linear system and condition (6) at  $\omega = 2$ .

### Non-Feedback Control methods

The following methods modify the underlying chaotic dynamical system weakly so that stable solutions appear [5].

**1- controlling of chaos by addition of constant torque:** Interestingly, one can even add just a constant torque to control or quench the chaotic attractor to a desired periodic one in a typical non-linear system. It ensures effective controlling in a very simple way. Examining the effect of constant torque, the added torque is assumed to be present in equation (4). In Fig.5, using constant torque control, chaotic behavior is modified to periodic motion for constant torque 4.

**2- controlling of chaos by the second periodic torque:** One can control chaotic systems by the addition of the external second period force in the chaotic state. Equation (4) with the second periodic torque  $b \sin(\Omega t)$  can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m_2} \left( m_3 \sin(x) - cy^3 - eyx^2 - \right) \\ \quad a \sin(\omega t) + b \sin(\Omega t) \end{cases}, \quad (8)$$

where  $b \in [10,15]$  and  $\Omega = 3.5$ , the detailed structure of Lyapunov exponent versus  $b$  is shown in Fig.6(a). Also, phase portraits and Poincaré map of equation (8) with the conditions (6),  $\Omega = 3.5$ , and  $b = 14$  (Fig. 6(b)) show a period 4-T motion.

### Time Delayed Feedback Control (TDFC)

In this section the application of delayed time feedback control [6] is introduced. The difference between the delayed output signal  $\beta(\tau - \tau_d)$  and the output signal  $\beta(\tau)$  is used as a control signal:

$$F(\tau) = K[\beta(\tau - \tau_d) - \beta(\tau)], \quad (9)$$

where  $K$  is the weight of control signal and  $\tau_d$  is the delay time. So, the equation (4) will be

$$\begin{cases} \dot{x} = y \\ \dot{y} = \frac{1}{m_2} \left( m_3 \sin(x) - cy^3 - eyx^2 - \right) \\ \quad a \sin(\omega t) + F(\tau) \end{cases} \quad (10)$$

Adjusting  $K$  and  $\tau_d$ , we can convert the chaotic motion to periodic motion or even quasi-periodic motion. Fig. 7, where  $K = 2$  and  $\tau_d = 12$  presents a periodic motion.

### Adaptive Control

Adaptive control algorithm was recently suggested [8] for multi-parameter and high dimensional non-linear systems. This control mechanism is remarkably effective in returning a system to its original dynamics after a sudden perturbation in the system parameters changes the dynamical behavior. The error signal governs the change of the parameter of the system, which readjusts so as to reduce the error to zero. For a general N-dimensional dynamical system

$$\dot{\beta} \equiv \frac{d\beta}{d\tau} = F(\beta, \tau, \mu), \quad (11)$$

where  $\beta \equiv (\beta_1, \beta_2, \dots, \beta_N)$  are variables and  $\mu \equiv (\mu_1, \mu_2, \dots, \mu_M)$  are parameters, which determine the nature of the dynamics. The prescription for effecting adaptive control is through the additional dynamics

$$\dot{\mu} = \zeta(\beta - \beta_s), \quad (12)$$

where  $\beta_s$  is the desired steady state value and  $\zeta$  indicates the stiffness of control.

This mechanism is remarkably effective and rapid, and is of utility in a large variety of systems, ranging from biological units to control engineering. Adaptive controlling can change chaos motion into periodic motion. The result is shown in Fig. 7. In Fig. 7(a) the parameter  $\mu = [a]$  and in Fig. 7(b) the parameter  $\mu = [a, b]^T$  is controlled.

### Conclusion

The dynamical system of the physical pendulum with damping subjected to excited torques exhibits a rich variety of non-linear behaviors as different parameters are varied. Due to effect of non-linearity, regular or chaotic motions may appear. In this paper, computational methods and controlling of chaos have been employed to study the dynamical behavior of this non-linear system.

The computational analyses have been performed. The bifurcation of the parameter-dependent system has been studied numerically. The time evolutions of non-linear dynamical system response have been describe using the phase portraits via the Poincaré

map technique. The occurrence and nature of chaotic attractors have been verified by evaluating Lyapunov exponent.

We have demonstrated that a simple control strategy can be effectively used to suppress chaos in a non-linear dynamical system. It is anticipated that similar control strategy can be successfully implemented for more complex situations.

### References

- 1- Ge, Z. M., Yang, C. H., Chen, H. H., and Lee, S. C., "Non-Linear Dynamics and Chaos Control of a Physical Pendulum with Vibrating and Rotating Support" *Journal of Sound and Vibration*, Vol.242, No.2, 2001, pp. 247-264.
- 2- Gwinn, E. G., and Westervelt, R. M., "Fractal Basin Boundary and Intermittency in the Driven Damped Pendulum" *Physical Review A*, Vol.33, 1986, pp. 4143-4155.
- 3- Wolf, A., Swift, J. B., Swinney, H. L., and Vastano, J. A., "Determining Lyapunov exponent from a time series" *Physica D*, Vol.16, 1985, pp. 285-317.
- 4- Spong, M. W., Corke, P., and Lorenzo, R., "Nonlinear Control of the Inertia Wheel Pendulum" *Automatica*, Vol.37, 1999, pp.1845-1851
- 5- Rajasekar, S., and Lakshmanan, M., "Algorithm for Controlling Chaotic Motion: Application for the BVP Oscillator" *Physica D*, Vol.67, 1993, pp. 282-300.
- 6- Pyragas, K., and Tamasevicius, A., "Experimental Control of Chaos by Delayed Self-Controlling Feedback" *Physics Letter A*, Vol.180, 1993, pp. 99-102.
- 7- G. Chen, *Controlling Chaos and Bifurcation in Engineering Systems*, CRC Press, 2000.
- 8- Sinha, S., Ramaswamy, R., and Rao, J. S., "Adaptive Control in Nonlinear Dynamics" *Physica D*, Vol.43, 1991, pp.118-128

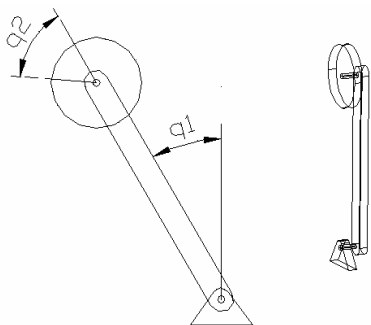


Fig. 1 – A schematic diagram of the pendulum.

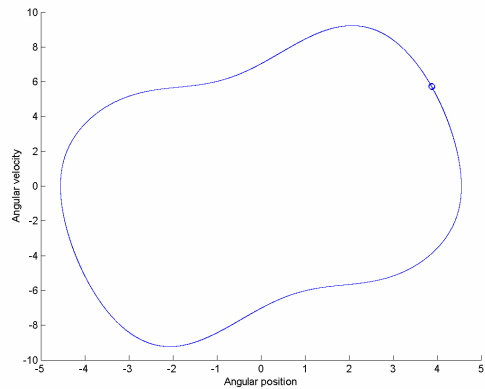


Fig. 2(a) – Poincaré maps superposed on phase portraits for  $\omega = 1.8$

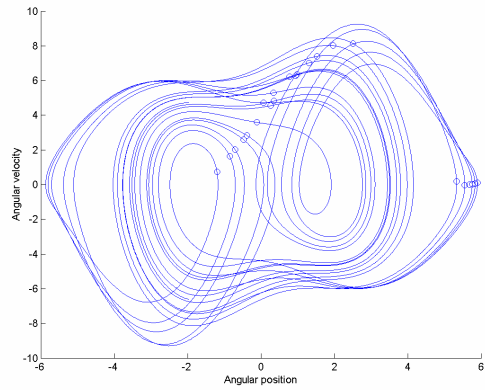


Fig. 2(b) – Poincaré maps superposed on phase portraits for  $\omega = 2.0$

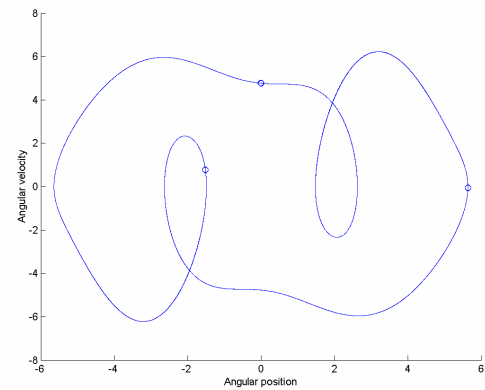
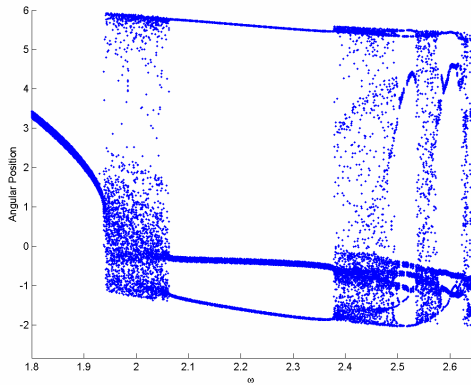
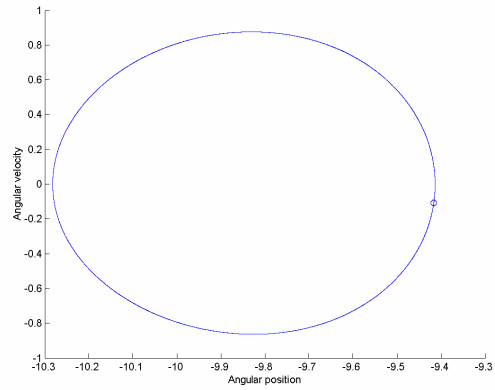


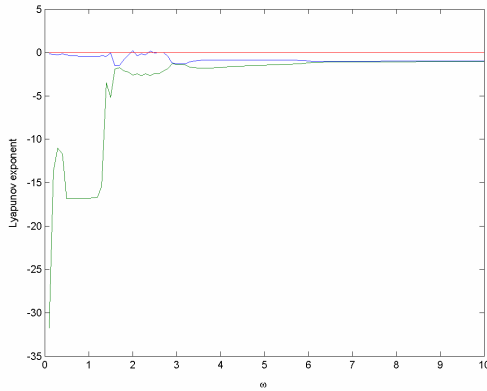
Fig. 2(c) – Poincaré maps superposed on phase portraits for  $\omega = 2.2$



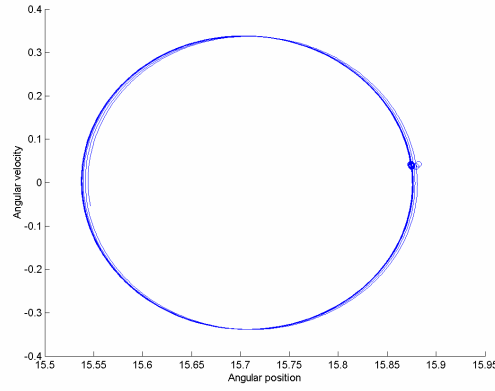
**Fig. 3 – Bifurcation diagram**



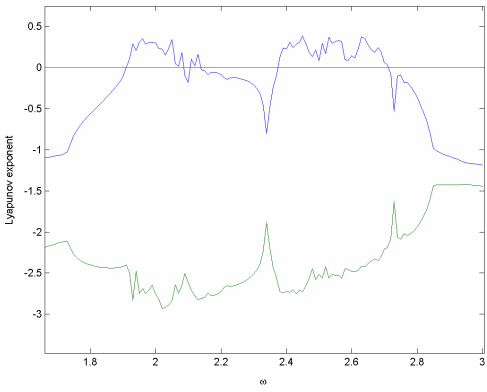
**Fig. 5 – Poincaré maps superposed on phase portraits of non-feedback of constant torque**



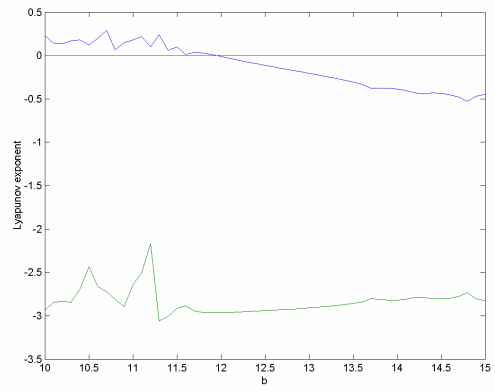
**Fig. 4(a) - Lyapunov exponent vs.  $\omega$**



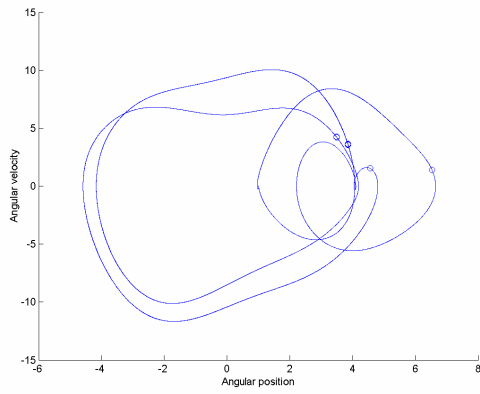
**Fig. 5 – Poincaré maps superposed on phase portraits of delayed-feedback**



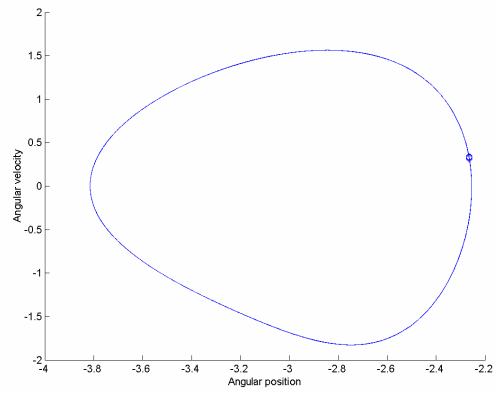
**Fig. 4(b) – Lyapunov exponent vs.  $\omega$**



**Fig. 6(a) – Lyapunov exponent diagram for non-feedback of the second periodic torque**



**Fig. 6(b)** – Poincaré maps superposed on phase portraits of non-feedback of the second periodic torque



**Fig. 7(b)** – Poincaré maps superposed on phase portraits of Adaptive control



**Fig. 7(a)** – Poincaré maps superposed on phase portraits of Adaptive control